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VELOCITY OF SOUND IN A MULTICOMPONENT MEDIUM AT REST

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The Kuropatenko model is considered, as applied to a multicomponent medium where the number of the sought functions coincides with the number of equations. The velocities of sound in a multicomponent medium at rest are determined. A formula of a polynomial of power N whose positive roots are squared velocities of sound in a medium with N components is derived. For $N = 2$, the values of two velocities of sound are determined in explicit form. It is demonstrated that the thus-found maximum value of the velocity of sound in a two-component medium containing nitrogen and oxygen with volume concentrations corresponding to air differs (in dimensionless form) from the velocity of sound in air by less than 0.3%. Numerical calculations predict the existence of three velocities of sound in a three-component medium. If the velocity of sound in all N components is identical, it is proved that the maximum velocity of sound in such a medium equals this velocity, and there is only one more velocity of sound in the medium, which has a lower value.

Key words: *multicomponent medium, sound characteristic, velocity of sound.*

A new mathematical model was proposed in [1] to describe the flows of multicomponent media. This model is a special quasi-linear system of equations with partial derivatives, which is based on conservation laws for the mixture obtained from conservation laws for individual components. Both the binary interactions of various components and the cluster interaction of the components with the virtual continuous medium are taken into account. One basic advantage of the Kuropatenko model (KM) is its closedness: the KM of a multicomponent medium contains an identical number of equations and functions, and its closure does not require additional hypotheses specifying the properties of the mixture.

In the present paper, we consider the KM of a multicomponent medium in the case of plane-symmetric flows, where each of the N components is an ideal polytropic gas. The main challenge of the paper was to derive an analytical expression for the velocity of sound in such a multicomponent medium at rest.

In the case considered, the KM written in dimensionless variables has the following form:

$$\frac{\partial \sigma_i}{\partial t} + u_i \frac{\partial \sigma_i}{\partial x} + \sigma_i \frac{\partial u_i}{\partial x} = 0; \tag{1}$$

$$\sigma_i \frac{\partial u_i}{\partial t} + \Big[(\gamma_i - 1)c_{vi}^0 T_i + (u - u_i)^2 \Big(\frac{\sigma_i}{\sigma} - \frac{1}{2}\Big)\Big] \frac{\partial \sigma_i}{\partial x}$$
$$+ \sigma_i \Big[u\Big(1 - \frac{\sigma_i}{\sigma}\Big) + \frac{\sigma_i}{\sigma} u_i\Big] \frac{\partial u_i}{\partial x} + (\gamma_i - 1)c_{vi}^0 \sigma_i \frac{\partial T_i}{\partial x} + \frac{(u - u_i)\sigma_i}{\sigma} \sum_{j=1, j\neq i}^{N} (u - u_j) \frac{\partial \sigma_j}{\partial x}$$
$$- \frac{(u - u_i)\sigma_i}{\sigma} \sum_{j=1, j\neq i}^{N} \sigma_j \frac{\partial u_j}{\partial x} - \alpha_i \sum_{j=1, j\neq i}^{N} \alpha_j a_{ji}(u_j - u_i) = 0; \tag{2}$$

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$$c^0_{vi}\sigma_i\,\frac{\partial T_i}{\partial t}+\frac{1}{2}\Big[\gamma_i c^0_{vi}(u-u_i)\Big(1-\frac{\sigma_i}{\sigma}\Big)T_i-u_i^2\Big]\frac{\partial\sigma_i}{\partial x}$$

$$+\,\sigma_i\Big\{c^0_{vi}\Big[\frac{1}{2}\,\gamma_i\Big(1+\frac{\sigma_i}{\sigma}\Big)-1\Big]T_i-\frac{3}{2}\,u_i^2-\frac{1}{2}\,(u-u_i)^2\Big\}\frac{\partial u_i}{\partial x}+c^0_{vi}\sigma_i\Big(u_i+\frac{1}{2}\,\gamma_i(u-u_i)\Big)\frac{\partial T_i}{\partial x}$$

$$-\,\frac{1}{2}\,\gamma_i c^0_{vi}\,\frac{\sigma_i T_i}{\sigma}\sum_{j=1,j\neq i}^{N}(u-u_j)\,\frac{\partial\sigma_j}{\partial x}+\frac{1}{2}\,\gamma_i c^0_{vi}\,\frac{\sigma_i T_i}{\sigma}\sum_{j=1,j\neq i}^{N}\sigma_j\,\frac{\partial u_j}{\partial x}$$

$$-\sum_{j=1,j\neq i}^{N}\Big\{\frac{1}{2}\,\alpha_i\alpha_j a_{ji}(u_j-u_i)^2+b_{ji}\Big[\alpha_i(\gamma_j-1)c^0_{vj}\sigma_j T_j-\alpha_j(\gamma_i-1)c^0_{vi}\sigma_i T_i\Big]+\alpha_i\alpha_j c_{ji}(T_j-T_i)\Big\}=0; \qquad (3)$$

$$p_i\,\frac{\partial\alpha_i}{\partial t}+c^0_{vi}\alpha_i(u-u_i)T_i\Big(\gamma_i\,\frac{\sigma_i}{\sigma}-1\Big)\frac{\partial\sigma_i}{\partial x}+\gamma_i c^0_{vi}\alpha_i\sigma_i T_i\Big(1-\frac{\sigma_i}{\sigma}\Big)\frac{\partial u_i}{\partial x}\frac{\partial T_i}{\partial x}+p_i u_i\,\frac{\partial\alpha_i}{\partial x}$$

$$+\,\gamma_i c^0_{vi}\,\frac{\alpha_i\sigma_i T_i}{\sigma}\sum_{j=1,j\neq i}^{N}(u-u_j)\,\frac{\partial\sigma_j}{\partial x}-\gamma_i c^0_{vi}\,\frac{\alpha_i\sigma_i T_i}{\sigma}\sum_{j=1,j\neq i}^{N}\sigma_j\,\frac{\partial u_j}{\partial x}=0. \qquad (4)$$

Here $1\le i\le N$, where $N\geqslant 2$ is the number of components in the multicomponent medium, i.e., the system contains $4N$ equations.

In Eqs. (1)–(4), the sought functions for each ith component are the partial density σ_i, velocity u_i, temperature T_i, and volume concentration α_i. The total number of the sought functions in system (1)–(4) is also $4N$. Thereby, we have $\sigma_i=\alpha_i\rho_i$, where ρ_i is the density of the ith component, and

$$\sigma=\sum_{j=1}^{N}\sigma_j,\qquad u=\frac{1}{\sigma}\sum_{i=1}^{N}\sigma_i u_i$$

are the partial density and velocity of the virtual medium, respectively.

The equations of state of each ith component are taken in the form

$$P_i=(\gamma_i-1)c^0_{vi}\rho_i T_i,\qquad E_i=c^0_{vi}T_i,\qquad p_i=\alpha_i P_i,\qquad i=1,2,\dots,N,$$

where P_i and E_i are the pressures and internal energies of the components and $\gamma_i=\mathrm{const}>1$ and $c^0_{vi}>0$ are the ratios of specific heats and the specific heats of the components, respectively. Then, the squared velocity of sound in each component is

$$c_i^2=\gamma_i(\gamma_i-1)c^0_{vi}T_i.$$

The constant positive exchange coefficients a_{ji}, b_{ji}, and c_{ji} are assumed to be given, and

$$a_{ji}=a_{ij},\quad b_{ji}=b_{ij},\quad c_{ji}=c_{ij},\quad 1\le i\le N,\quad 1\le j\le N.$$

Before studying the issue of the velocity of sound in a multicomponent medium at rest, i.e., the existence of the corresponding characteristics of system (1)–(4), we should given one comment of principal importance.

In deriving the KM of a multicomponent medium in [1], as well as in deriving other models of heterogeneous media (see, e.g., [2]), the following equality is postulated to be valid:

$$\sum_{i=1}^{N}\alpha_i=1. \qquad (5)$$

Both in [1] and in the present paper, however, equality (5) is not present explicitly. We do not consider this equality specially in the present paper, which allows us to study a system containing only differential but not functional equations. Moreover, if relation (5) is added to system (1)–(4), the system becomes overdetermined: the number of equations becomes greater than the number of the sought quantities. A further analysis of this overdetermined system is rather difficult.

Naturally, the mere fact of including or not including condition (5) into the system of the KM of a multicomponent medium also exerts a significant effect on the number of characteristics and on the values of velocities along these characteristics.

Two options are possible to justify the fact that relation (5) is not included into system (1)–(4): 1) to prove that relation (5) is a corollary of system (1)–(4); 2) to check the validity of equality (5) for each solution of system (1)–(4) obtained.

In the first case, we obtain the following theorem.

Theorem 1. *If for the described KM of a multicomponent medium* (1)–(4) *of flows equilibrium in terms of velocities, i.e., in the case*

$$u_i(t,x) = u_j(t,x) = u(t,x), \qquad 1 \le i \le N, \quad 1 \le j \le N, \tag{6}$$

the equality

$$\sum_{i=1}^{N} \alpha_i(t,x)\Big|_{t=0} = 1 \tag{7}$$

is valid, then equality (5) *holds for all t and x for which $u_i(t,x)$ are continuous and $p_i(t,x) \ne 0$, $1 \le i \le N$.*

Proof. If condition (6) is valid and the inequalities $p_i(t,x) \ne 0$ and $1 \le i \le N$ are satisfied, all Eqs. (4) transform to the transport equations

$$\frac{\partial \alpha_i}{\partial t} + u(t,x)\,\frac{\partial \alpha_i}{\partial x} = 0, \qquad 1 \le i \le N,$$

whose summation yields a similar equation for the function $\alpha(t,x) = \sum\limits_{i=1}^{N} \alpha_i(t,x)$:

$$\frac{\partial \alpha}{\partial t} + u(t,x)\,\frac{\partial \alpha}{\partial x} = 0. \tag{8}$$

Condition (7) is the initial condition for Eq. (8) and ensures solution uniqueness. The existence of the solution of the transport equation (8) ensures continuity of the function $u(t,x)$. This unique solution of Eq. (8) is the function

$$\alpha(t,x) = 1.$$

Theorem 1 is proved.

Proving a similar theorem in the general case is a difficult task.

In the present work, we do not construct sophisticated solutions of system (1)–(4). We study only one property of this system by an example of a particular simple solution for which the above-proved theorem is valid. Therefore, we do not need to justify the second variant of non-including equality (5) into system (1)–(4) in the case of other exact solutions.

System (1)–(4) has an exact solution

$$\begin{gathered} \sigma_i = \sigma_i^0 = \text{const} > 0, \quad u_i = 0, \quad T_i = T_i^0 = \text{const} > 0, \\ \alpha_i = \alpha_i^0 = \text{const} > 0, \quad 1 \le i \le N, \end{gathered} \tag{9}$$

which describes the homogeneous rest where all components are in equilibrium not only in terms of velocities (all velocities are equal to zero) but also in terms of temperatures and pressures:

$$T_i^0 = 1, \qquad P_i^0 = P^0 = \text{const} > 0.$$

To construct such a solution, we first choose the constants α_i^0 to satisfy the equality

$$\sum_{i=1}^{N} \alpha_i^0 = 1.$$

Then, we have to choose the constants ρ_i^0 so that the pressures of all components are equal, for instance, to unity:

$$1 = P^0 = (\gamma_i - 1) c_{vi}^0 \rho_i^0, \qquad 1 \le i \le N.$$

After that, we can unambiguously determine the constants ρ_i^0:

$$\rho_i^0 = 1/[(\gamma_i - 1) c_{vi}^0], \qquad 1 \le i \le N.$$

As $\sigma_i^0 = \alpha_i^0 \rho_i^0$, unambiguous determining of the constant ρ_i^0 leads to unambiguous determining of the constants σ_i^0:

$$\sigma_i^0 = \alpha_i^0/[(\gamma_i - 1) c_{vi}^0], \qquad 1 \le i \le N.$$

The resultant exact solution (9) of system (1)–(4), which is called the homogeneous equilibrium rest, is denoted as $\boldsymbol{U}_i^0$, $1 \le i \le N$.

It is known that the characteristics of a quasi-linear system are determined only on a particular specified solution [3]. The characteristics of system (1)–(4) are further constructed on the homogeneous equilibrium rest (9). For this purpose, the values of the gas-dynamic parameters (9) are substituted to the coefficients at the partial derivatives in system (1)–(4) considered. To simplify calculations, we make the coefficients at the derivatives with respect to time in the resultant equations equal to unity by dividing by appropriate constants. In addition, we take into account that $(c_i^0)^2 = \gamma_i(\gamma_i - 1)c_{vi}^0$ and introduce the notation $\theta_i^0 = (c_i^0)^2$, $\delta_i^0 = \sigma_i^0/\sigma^0$. As a result, we obtain the following expression for the principal part of the system:

$$\boldsymbol{U}_t + G\boldsymbol{U}_x.$$

The vector of unknown functions $\boldsymbol{U}$ here contains $4N$ components: σ_1, u_1, T_1, α_1, $\ldots$, σ_N, u_N, T_N, and α_N; the matrix G has the dimension $4N \times 4N$.

The characteristics of system (1)–(4) on solution (9) are straight lines:

$$x - x_0 = k(t - t_0) \tag{10}$$

(x_0 and t_0 are constants and k are the velocities of propagation of small perturbations called the velocities of sound in gas dynamics [3]). The straight line (10) is the characteristic on solution (9) if and only if the value k is the root of the characteristic equation

$$\Delta_N = \det(G - kE) = 0. \tag{11}$$

Let us first consider the case $N = 2$, i.e., the case of a two-component medium. The determinant Δ_2 is a determinant of the eighth order:

$$\Delta_2 = \begin{vmatrix} -k & \sigma_1^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{21} & -k & g_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{32} & -k & 0 & 0 & g_{36} & 0 & 0 \\ 0 & g_{42} & 0 & -k & 0 & g_{46} & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & \sigma_2^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{65} & -k & g_{67} & 0 \\ 0 & g_{72} & 0 & 0 & 0 & g_{76} & -k & 0 \\ 0 & g_{82} & 0 & 0 & 0 & g_{86} & 0 & -k \end{vmatrix}.$$

Here

$$g_{21} = \frac{\theta_1^0}{\gamma_1 \sigma_1^0}, \quad g_{23} = \frac{\theta_1^0}{\gamma_1}, \quad g_{32} = \frac{\gamma_1(1 + \delta_1^0)}{2} - 1, \quad g_{36} = \frac{\gamma_1 \delta_2^0}{2},$$

$$g_{42} = \frac{\theta_1^0 \sigma_1^0 (1 - \delta_1^0)}{\gamma_1 - 1}, \quad g_{46} = -\frac{\theta_1^0 \sigma_1^0 \delta_2^0}{\gamma_1 - 1}, \quad g_{65} = \frac{\theta_2^0}{\gamma_2 \sigma_2^0}, \quad g_{67} = \frac{\theta_2^0}{\gamma_2},$$

$$g_{72} = \frac{\gamma_2 \delta_1^0}{2}, \quad g_{76} = \frac{\gamma_2(1 + \delta_2^0)}{2} - 1, \quad g_{82} = -\frac{\theta_2^0 \sigma_2^0 \delta_1^0}{\gamma_2 - 1}, \quad g_{86} = \frac{\theta_2^0 \sigma_2^0 (1 - \delta_2^0)}{\gamma_2 - 1}.$$

By applying simple identity transformations, we obtain the following presentation of the sought determinant Δ_2 via the fourth-order determinant:

$$\Delta_2 = k^4 \begin{vmatrix} -k & g_{23} & 0 & 0 \\ g_{32} + 1 & -k & g_{36} & 0 \\ 0 & 0 & -k & g_{67} \\ g_{72} & 0 & g_{76} + 1 & -k \end{vmatrix}.$$

This determinant is decomposed over its first row:

$$\Delta_2 = k^4\{k^4 - [(1 + \delta_1^0)\theta_1^0 + (1 + \delta_2^0)\theta_2^0]k^2/2 + (1 + \delta_1^0 + \delta_2^0)\theta_1^0\theta_2^0/4\} \equiv k^4 P_2(k^2).$$

The fourth-power polynomial of k in braces is a biquadratic polynomial of $\lambda = k^2$. With allowance for $\delta_1^0 + \delta_2^0 = 1$, the roots of this polynomial $P_2(\lambda)$ are determined by the formula

$$\lambda_\pm = [(1+\delta_1^0)\theta_1^0 + (1+\delta_2^0)\theta_2^0]/4 \pm \sqrt{[(1+\delta_1^0)\theta_1^0 + (1+\delta_2^0)\theta_2^0]^2 - 8\theta_1^0\theta_2^0}/4.$$

As the radicand

$$[(1+\delta_1^0)\theta_1^0 + (1+\delta_2^0)\theta_2^0]^2 - 8\theta_1^0\theta_2^0 = [(1+\delta_1^0)\theta_1^0 - (1+\delta_2^0)\theta_2^0]^2 + 4\delta_1^0\delta_2^0\theta_1^0\theta_2^0$$

is positive for $\delta_i^0 > 0$ and $\theta_i^0 > 0$ $(i = 1, 2)$, both roots $\lambda_\pm$ are also positive, and the polynomial $P_2(k^2)$ has four real roots.

Hence, for $N = 2$, the characteristic equation (11) $\Delta_2 = 0$ has eight real roots with allowance for their multiplicity.

The first root of the characteristic equation is the zero quadruple root

$$k_{1,2,3,4} = 0$$

with the corresponding contact characteristic of multiplicity equal to four in the two-component medium. The next four roots of the characteristic polynomial are described by the formulas

$$k_{5,6} = \pm\sqrt{\lambda_+}, \qquad k_{7,8} = \pm\sqrt{\lambda_-}.$$

Four sound characteristics (each of multiplicity equal to one) correspond to these roots. The sonic characteristics of the two-component medium propagate to the right and to the left with two velocities of sound $c_h > c_l$:

$$c_h = \left\{(1+\delta_1^0)\theta_1^0 + (1+\delta_2^0)\theta_2^0 + \sqrt{[(1+\delta_1^0)\theta_1^0 - (1+\delta_2^0)\theta_2^0]^2 + 4\delta_1^0\delta_2^0\theta_1^0\theta_2^0}\right\}^{1/2}/2; \qquad (12)$$

$$c_l = \left\{(1+\delta_1^0)\theta_1^0 + (1+\delta_2^0)\theta_2^0 - \sqrt{[(1+\delta_1^0)\theta_1^0 - (1+\delta_2^0)\theta_2^0]^2 + 4\delta_1^0\delta_2^0\theta_1^0\theta_2^0}\right\}^{1/2}/2 \qquad (13)$$

(c_h is the higher velocity of sound in the multicomponent medium at rest and c_l is the lower velocity of sound in the same medium).

The presence of four sound characteristics in the case of two-component media can be readily explained. In its form, system (1)–(4) is close to systems of multivelocity models of multicomponent media, which also have four sound characteristics propagating in pairs in different directions in the case of a two-component medium [4].

If we assume for certainty but without losing generality that the second component has a higher velocity of sound $c_2^0 > c_1^0$ (otherwise, the components are numbered in the opposite manner), we obtain the following inequalities from Eqs. (12) and (13):

$$c_2^0 > c_h > c_1^0 > c_l. \qquad (14)$$

It seems that inequalities (14) are physically meaningful results of using the KM of a multicomponent medium (1)–(4). Indeed, in addition to elastic interaction of particles of the medium, the model considered takes into account exchange of momentum and energy between particles of different components. Such an interaction reduces the fraction of energy spent during the elastic interaction. As it is the elastic interaction that determines the velocity of disturbance propagation in the medium, allowance for additional interactions leads to lower velocities of sound in a multicomponent medium than the velocities of sound in each component.

Let us compare the velocity of sound in air obtained in physical experiments [5, 6] and the value calculated by Eq. (12). It is known that the volume fractions of nitrogen N_2 and oxygen O_2 in air are 78.08 and 20.95%, respectively, [5, 6]. The remaining 0.97% are the inert gases and carbon dioxide. We assume that air is a two-component medium and distribute the remaining 0.97% in proportion between nitrogen and oxygen, i.e., the volume concentrations of nitrogen and oxygen are assumed to be 0.7886 and 0.2114, respectively. Under standard conditions (with the pressure equal to 1 atm and the temperature equal to 16.7°C), the densities of nitrogen and oxygen are 1.25046 and 1.42897 kg/m^3. The velocities of sound in nitrogen and oxygen are 347.6 and 323.8 m/sec. Substituting these values into Eqs. (12) and (13), we obtain $c_h = 342.697429$ and $c_l = 232.236655$. The velocities of sound in air were given in [6]: $c_a = 331.8$, 337.8, and 343.8 m/sec at $T = 0$, 10, and 20°C, respectively. Determining the value of c_a at $T = 16.7$°C by linear interpolation, we obtain $c_a = 341.82$. The difference in the relative values of c_h and c_a $(|c_a - c_h|/c_a) \cdot 100\%$ does not exceed 0.3%. Thus, for air considered as a two-component medium, the KM of a multicomponent medium is adequate to results of physical experiments on determining the velocity of sound.

Remark 1. Usually air is mentioned as an example of homogeneous media where volume concentrations are not used, because the gases in air are assumed to be mixed at the molecular level, i.e., the volumes occupied by each gas in the multicomponent mixture are identical: $V_1 = \ldots = V_N = V$ [7]. Nevertheless, the assumption that air is a heterogeneous medium with $V_1 + \ldots + V_N = V$ [7] does not seem to be contradictory. The point is that the exchange of momentum and energy between various components of a heterogeneous medium is determined by the exchange coefficients a_{ij}, b_{ij}, and c_{ij}. These quantities, however, are not used in calculating the velocities of propagation of characteristics. Moreover, numerous experiments with air made it possible to determine the volume concentrations α_i for each gas contained in air. This fact allows us to use Eqs. (12) and (13) to evaluate the adequacy of the KM of a multicomponent medium to the corresponding results of physical experiments.

Remark 2. For heterogeneous models of two-component media in the case of their equilibrium in terms of velocities and pressures, we can determine the equilibrium velocity of sound (Wood's velocity) [2]:

$$c = 1\Big/\sqrt{\rho\Big(\frac{\alpha_1}{\rho_1 c_1^2} + \frac{\alpha_2}{\rho_2 c_2^2}\Big)}, \qquad \rho = \alpha_1\rho_1 + \alpha_2\rho_2.$$

For the above-indicated parameters for air, this formula yields $c = 342.17$ m/sec, which is closer to the experimental value of c_a than the calculated value c_h. It should be borne in mind, however, that Wood's formula for velocity was obtained with the use of the derived equation of state of the entire multicomponent medium in the form $p = p(\rho, T, \alpha_1)$ and under the assumption that the motion of an equilibrium two-phase mixture is described by conventional equations of a single-phase continuous medium [2], i.e., $c = \sqrt{\partial p/\partial \rho}\,\Big|_{S=\mathrm{const}}$, where S is the entropy. In contrast to Wood's formula for velocity, the formulas for c_h and c_l were obtained through analyzing the entire system (1)–(4), which, naturally, differs from the traditional system of gas-dynamic equations for a one-component medium.

The following factors are observed for Eqs. (12) and (13). Let both components have identical velocities of sound

$$c_1^0 = c_2^0 = c^0; \tag{15}$$

hence, we have $\theta_1^0 = \theta_2^0$. In particular, Eq. (15) can follow from the identical thermodynamic parameters of the components:

$$\gamma_1 = \gamma_2, \qquad c_{v1}^0 = c_{v2}^0.$$

Then, we obtain the following expressions for the velocities of sound in a two-component medium, independent of particular values of the volume concentrations of each component, i.e., independent of the values of $\delta_{1,2}^0 = \sigma_{1,2}^0/\sigma^0$ ($\delta_1^0 + \delta_2^0 = 1$):

$$c_h = c^0, \qquad c_l = c^0/\sqrt{2}. \tag{16}$$

The coincidence of the higher velocity of sound with the general velocity of sound in a particular case of identical velocities of sound of both components and the mixture as a whole seems to be an expected physical property.

The fact that there is one more velocity of sound in a two-component medium with identical velocities of sound of both components, which differs from the general velocity of sound and is rigorously lower than the latter, is apparently a consequence of both mathematical and physical facts.

The equality of the velocities of sound does not necessarily imply the equality $\boldsymbol{U}_1 = \boldsymbol{U}_2$, i.e., identical thermodynamic parameters of the components do not necessarily imply identical gas-dynamic parameters of the components. The limit transition from $\boldsymbol{U}_1 \neq \boldsymbol{U}_2$ to $\boldsymbol{U}_1 = \boldsymbol{U}_2$ within the framework of equations with partial derivatives is not "continuous," which follows from the model example considered below.

Let the coefficients $a_i > b_i > 0$ ($i = 1, 2$) in the system

$$\begin{aligned} &v_t + a_1 v_x + b_1 w_x = 0,\\ &w_t + a_2 w_x + b_2 v_x = 0 \end{aligned} \tag{17}$$

be constant. Then, system (17) has two characteristics propagating with the velocities

$$c_{h,l} = [a_1 + a_2 \pm \sqrt{(a_1 - a_2)^2 + 4b_1 b_2}\,]/2.$$

If we assume that

$$a_1 = a_2 = a, \qquad b_1 = b_2 = b$$

in the last formula, we obtain

$$c_h = a + b > a - b = c_l,$$

i.e., there are still two velocities of sound even if the corresponding coefficients of system (17) are equal to each other.

If we assume that $v = w$ in system (17), it is split into two independent equations

$$v_t + (a_1 + b_1)v_x = 0, \qquad w_t + (a_2 + b_2)w_x = 0,$$

each having one characteristic with the following velocities of propagation:

$$c_1 = a_1 + b_1, \qquad c_2 = a_2 + b_2.$$

For $a_1 = a_2 = a$ and $b_1 = b_2 = b$, these velocities are identical: $c_1 = c_2 = a + b$.

If we set $\boldsymbol{U}_1 = \boldsymbol{U}_2$ in system (1)–(4) with $N = 2$, by analogy with the procedure described above, then system (1)–(4) for each $i = 1, 2$ is split into two subsystems equivalent to the system of equations of gas dynamics. If we assume that thermodynamic parameters are identical, in addition to the equality of gas-dynamic parameters, the velocities of sound determined by these two systems of gas-dynamic equations are identical.

One possible physical reason for validity of Eqs. (15) and (16) can be described as follows. For instance, the thermodynamic parameters of a two-component medium consisting of water and water vapor are identical. Yet, the components can exchange their momentum and energy. It is this interaction that is taken into account by the KM of a multicomponent medium. It is because of the interaction of different components and energy spent on this interaction that the gas-dynamic parameters of different components are not identical. Therefore, we have to consider the entire system (1)–(4) with $N = 2$, which has two velocities of sound $c_h > c_l$ in the homogeneous state at rest, rather than two individual systems of gas-dynamic equations.

In the case of three-component and four-component media, we obtain the following formula after finding the determinants Δ_3 and Δ_4:

$$\Delta_N = k^{2N} P_N(k^2). \tag{18}$$

For $\lambda = k^2$, the polynomial $P_N(\lambda)$ in this formula takes the form

$$P_N(\lambda) = \lambda^N - \frac{\lambda^{N-1}}{2}\sum_{i=1}^{N}(1+\delta_i^0)\theta_i^0 + \frac{\lambda^{N-2}}{4}\sum_{\substack{i,j=1\\ i<j}}^{N}(1+\delta_i^0+\delta_j^0)\theta_i^0\theta_j^0 - \frac{\lambda^{N-3}}{8}\sum_{\substack{i,j,m=1\\ i<j<m}}^{N}(1+\delta_i^0+\delta_j^0+\delta_m^0)\theta_i^0\theta_j^0\theta_m^0$$

$$\ldots + (-1)^l\,\frac{\lambda^{N-l}}{2^l}\sum_{\substack{i_1,\ldots,i_l=1\\ i_1<\ldots<i_l}}^{N}\Bigl(1+\sum_{j=1}^{l}\delta_{i_j}^0\Bigr)\prod_{j=1}^{l}\theta_{i_j}^0 + \ldots + (-1)^N\,\frac{1}{2^N}\Bigl(1+\sum_{i=1}^{N}\delta_i^0\Bigr)\prod_{i=1}^{N}\theta_i^0. \tag{19}$$

With allowance for $\delta_1^0 + \ldots + \delta_N^0 = 1$, the last term in Eq. (19) is

$$\frac{(-1)^N}{2^{N-1}}\prod_{i=1}^{N}\theta_i^0.$$

For an arbitrary value of N, Eqs. (18) and (19) are proved by induction; the determinants Δ_N for an arbitrary value of $N \geq 3$ are calculated by the same scheme as in the case with $N = 2$.

The roots of the characteristic polynomial

$$\Delta_N \equiv k^{2N} P_N(k^2)$$

of power $4N$ with $N \geq 3$ include a zero root of multiplicity $2N$, which corresponds to a contact characteristic of the same multiplicity. The remaining roots of the characteristic polynomial Δ_N are such that their squares are roots of the polynomial $P_N(\lambda)$ of power N ($\lambda = k^2$).

For $N \geq 3$, it is rather difficult to prove that all roots of the polynomial $P_N(\lambda)$ are real positive numbers.

For $N = 3$, Cardano's formulas were assumed to hold for the roots of the cubic polynomial in [8]; moreover, it was proved that all roots are real if the following inequality is valid:

$$q^2/4 + p^3/27 < 0. \tag{20}$$

In Eq. (20), we have

$$p = -\frac{b^2}{3} + c, \quad q = \frac{2b^3}{27} - \frac{bc}{3} + d, \quad b = -\frac{1}{2}\sum_{i=1}^{3}(1+\delta_i^0)\theta_i^0, \quad d = -\frac{1}{4}\,\theta_1^0\theta_2^0\theta_3^0,$$

$$c = [(1+\delta_1^0+\delta_2^0)\theta_1^0\theta_2^0 + (1+\delta_1^0+\delta_3^0)\theta_1^0\theta_3^0 + (1+\delta_2^0+\delta_3^0)\theta_2^0\theta_3^0]/4.$$

Based on the validity of Eq. (20), we can easily prove that all roots of the polynomial $P_3(\lambda)$ are positive and, hence, the polynomial $P_3(k^2)$ has six real roots.

For arbitrary values of

$$\delta_i^0 > 0, \quad \theta_i^0 > 0, \quad i = 1, 2, 3, \qquad \delta_1^0 + \delta_2^0 + \delta_3^0 = 1$$

it is difficult to prove the validity of inequality (20), in particular, because the value of $q^2/4 + p^3/27$ tends to zero with a sixth order as $\theta_i^0 \to +0$. The left side of inequality (20) also tends to zero with the same order as $\delta_i^0 \to +0$.

At the moment, it is not proved that the polynomial $P_N(k^2)$ with $N \geq 3$ has $2N$ real roots and there are ultimately no formulas for calculating the roots in the case with $N \geq 5$. Nevertheless, the current level of development of computational tools and advanced software allow the roots of polynomial (19) to be found numerically, rather easily and with a required accuracy, for all values of N used in practice with prescribed values of δ_i^0 and θ_i^0. In particular, the validity of inequality (20) was established with allowance for the equality $\delta_1^0+\delta_2^0+\delta_3^0 = 1$ by means of an independent direct search of the values of the parameters δ_i^0 and $\mu_i^0 = \theta_i^0/\theta_3^0$ $(i = 1, 2)$ from 0.01 to 0.99 with a step of 0.01 under the assumption that $\theta_3^0 \geq \theta_2^0$, $\theta_3^0 \geq \theta_1^0$, and $0 < \delta_1^0 + \delta_2^0 < 1$. Thus, the existence of sound characteristics with three velocities of propagation of each of them was numerically confirmed for a three-component medium.

If the velocities of sound of all N components are identical

$$c_i^0 = c^0 > 0, \qquad 1 \leq i \leq N, \tag{21}$$

i.e., if $\theta_i^0 = \theta^0 = (c^0)^2 > 0$, $i = 1, \ldots, N$, we can easily prove the following equality with the use of Eq. (19):

$$P_N(\lambda) = (\lambda - \theta^0)(\lambda - \theta^0/2)^{N-1}. \tag{22}$$

Thus, if relations (21) hold, the polynomial $P_N(\lambda)$ is presented in the form of N real multipliers, both roots of the polynomial $\lambda_1 = \theta^0$ and $\lambda_2 = \theta^0/2$ being positive numbers.

It follows from Eq. (22) that the velocity c_h equals the general velocity of sound of all components in the case of Eq. (21), as in the case with $N = 2$ [see Eqs. (15) and (16)]:

$$c_h = c^0. \tag{23}$$

The multiplicities of the corresponding sound characteristics are equal to unity.

In the case with Eq. (21), there is only one velocity c_l:

$$c_l = c^0/\sqrt{2}, \tag{24}$$

and the multiplicities of the corresponding sound characteristics are equal to $N - 1$.

Thus, in the case with Eq. (21), formulas (22)–(24) generalize Eqs. (16) to the case with an arbitrary $N \geq 3$.

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